Relations between Adiabatic and Incompressible (Non-Adiabatic) Systems and Their Stability

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We examine in general whether the results valid for an incompressible, non-adiabatic system can be deduced from the results valid for an adiabatic (or a more general) system. A simple rule will be established by which an energy principle for incompressible, non-adiabatic perturbations is obtained from the energy principle for adiabatic perturbations. Application yields in particular the energy principles for magnetodynamic, respectively gravitational, respectively gravitational and magnetodynamic stability for incompressible, non-adiabatic perturbations which are the analogues of the energy principles of Bernstein et al., respectively Chandrasekhar, respectively Krüger and Callebaut for adiabatic perturbations.

It is proved that an equilibrium state is more stable or at least equally stable for incompressible, non-adiabatic perturbations than for adiabatic ones. The conditions under which the adiabatic regime and the incompressible one are both stable or both unstable are studied.

More detailed comparison theorems are enunciated for the case of magnetodynamic stability and all cases where the energy integral for $\gamma=0$ is independent of $\xi_{||}$ the component of $\boldsymbol{\xi}$ parallel to $\boldsymbol{B}$. If div $\boldsymbol{\xi}$ can be chosen arbitrarily when $\boldsymbol{\xi}_{\perp}$ is given then the adiabatic and the incompressible regimes are both stable or both unstable. A detailed examination whether div $\boldsymbol{\xi}$ can be chosen arbitrarily or not due to the presence of closed field lines leads to a classification of the perturbations in two cases. We compare the stability between these two cases for the adiabatic regime and for the incompressible one and for each case, the stability between both regimes. A similar analysis is given for restrictive conditions on div $\boldsymbol{\xi}$ due to the presence of closed pressure shells. In general only one case of the complete classification has to be considered to decide on the stability. Moreover the adiabatic and incompressible regimes are both stable or both unstable for most infinitely long tubes. The whole treatment is illustrated by the example of the linear pinch.

The macroscopic description of a multiparticle system is usually based on the conservation of mass, momentum, energy, magnetic flux and entropy [1] (adiabatic system). For an incompressible fluid, the conservation of entropy is replaced by the conservation of mass density (incompressible, non-adiabatic system). Either one of the descriptions may be used according to the chosen model of the physical situation. The adiabatic regime (A) is sometimes replaced by the incompressible one (regime I) for reasons of mathematical simplicity. It is of interest to know the relations between both descriptions and their stability.

It is well-known that an energy principle can be developed for regime A. Whether this is possible for the regime I and whether the energy principle for regime I follows from the one for A is not a matter of course and will be examined carefully. The analysis leads to a comparative study between adiabatic and incompressible, non-adiabatic perturbations, with emphasis on their stability properties.

The paper is arranged as follows. In section 1 we examine in a general way whether the results valid for an incompressible, non-adiabatic system can be deduced from the results valid for an adiabatic (or a more general) system. In section 2, it is proved that an energy principle for incompressible, non-adiabatic perturbations can be obtained from an energy principle for adiabatic perturbations in a simple way (theorem 1). Application yields in particular the energy principles for magnetodynamic, respectively gravitational, respectively gravitational and magnetodynamic stability for incompressible, non-adiabatic perturbations which are the analogues of the energy principles of BERNSTEIN et al. [2], respectively CHANDRASEKHAR [3], respectively KRÜGER [4, 5] and CALLEBAUT [5] and KOVETZ [6]. In section 3 we compare the stability properties of an

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equilibrium state, now governed by incompressible, non-adiabatic perturbations, now by adiabatic ones. It is proved that an equilibrium state is more stable or at least equally stable for incompressible, non-adiabatic perturbations than for adiabatic ones (theorem 2). Conditions under which the regimes A and I are both stable or both unstable are studied (theorem 3).

Section 4 deals in more detail with specific energy integrals, in particular the energy integral for magnetodynamic stability and more generally any energy integral which for $\gamma = 0$ is independent of $\xi_{\parallel}$, the component of $\boldsymbol{\xi}$ parallel to $\boldsymbol{B}$. According to theorem 3, if div $\boldsymbol{\xi}$ can be chosen arbitrarily when $\boldsymbol{\xi}_{\perp}$ is given, then the adiabatic and the incompressible regimes are both stable or both unstable. In section 4.1 we investigate in detail for the adiabatic regime, whether div $\boldsymbol{\xi}$ can be chosen arbitrarily or not according to the constraint $\langle s \rangle_{\text{tube}} = 0$ due to the presence of closed field lines. Here $s \equiv \operatorname{div} \boldsymbol{\xi} - \operatorname{div} \boldsymbol{\xi}_{\perp}$ and $\langle s \rangle_{\text{tube}}$ indicates the mean value of s over a closed flux tube. This leads to a classification of the perturbations in two cases, each corresponding to an energy integral with a different structure. If the parameters characterizing each case are infinitely close to each other there appears a discontinuity between the energy integrals, leading to comparison theorems for the adiabatic regime. The same stability analysis is done for the discontinuity between the energy integrals of the incompressible regime. Finally we compare, for each of the two cases, the stability between both regimes (theorem 4).

In section 4.2 the perturbations are further classified and compared according to the constraint $\langle s \rangle_{\text{shell}} = 0$, due to the presence of closed pressure shells. $\langle s \rangle_{\text{shell}}$ indicates now the mean value of s over a closed pressure shell.

In section 4.3 we illustrate the existence of the different energy integrals, the related discontinuities and the related stability properties by the example of the linear pinch (theorem 5).

1. Comparison of Adiabatic and Incompressible Formalisms

The utmost amount of information which can be deduced from a statistical theory without defining the form of the velocity distribution is to be found in the zeroth, the first and the second moment equations, these being respectively the equation of continuity (1), the equation of motion (2), and the equation of state (3):

$$\frac{d\varrho}{dt} + \varrho \operatorname{div} \boldsymbol{v} = 0, \tag{1}$$

$$\varrho \frac{d\boldsymbol{v}}{dt} + \operatorname{grad} p - \varrho \boldsymbol{f} = 0, \tag{2}$$

$$\frac{dp}{dt} + \gamma p \operatorname{div} \boldsymbol{v} + (\gamma - 1) \operatorname{div} \boldsymbol{q} = 0. \tag{3}$$

ϱ stands for the mass density, $\boldsymbol{v}$ for the macroscopic velocity, $\boldsymbol{f}$ for the force per unit of mass, p for the pressure (the pressure tensor is supposed to be isotropic), $\boldsymbol{q}$ for the thermal flux vector and γ for the ratio of specific heats. Field equations (e. g. the Maxwell equations and/or gravitational equations) which we will call Eq. (4), are added to the Eqs. (1) – (3). To keep the discussion as general as possible, the fields on which the force depend are not explicitly given but they are supposed to be independent of $\boldsymbol{q}$.

The system (1) – (4) is not a closed system. To make it closed, an additional supposition must be introduced. For this fifth equation we take

$$\frac{dp}{dt} - \Gamma \frac{p}{\varrho} \frac{d\varrho}{dt} = 0 \tag{5}$$

where Γ is a prescribed function. Eq. (5) combined with Eqs. (1) and (3) leads to

$$(\gamma - 1) \operatorname{div} \boldsymbol{q} + p(\gamma - \Gamma) \operatorname{div} \boldsymbol{v} = 0 \tag{6}$$

which shows that Eq. (5) is the most general equation closing the system, provided that Γ is an arbitrary function. Indeed, since the heat flow appears only in div $\boldsymbol{q}$ in the Eqs. (1) – (4), it suffices to express div $\boldsymbol{q}$ in the other variables in order to make the system closed.

Putting $\Gamma = \gamma$ (constant) yields the conservation of entropy density

$$\frac{d}{dt} (p \varrho^{-\gamma}) = 0 \tag{7}$$

and Eq. (6) leads to the adiabatic law

$$\operatorname{div} \boldsymbol{q} = 0. \tag{8}$$

Putting $\Gamma = \infty$ (or letting $\Gamma \to \infty$ from any particular law) and assuming dp/dt to be finite, yields the conservation of mass density (incompressibility)

$$d\varrho/dt = 0 \tag{9}$$

and Eq. (6) yields

$$\operatorname{div} \boldsymbol{v} = 0. \tag{10}$$

Other prescriptions for Γ (e. g. $\Gamma = \text{constant}$ and $\Gamma = \gamma + p_0/p$) may reveal physical interesting systems which are non-adiabatic and compressible [see Eq. (6)]. A large class of these is given by the cases where p is a function of ϱ only: $p = f(\varrho)$. We have then $\Gamma = f'(\varrho)\,\varrho/p$.

For every prescribed function Γ, the Eqs. (1), (2), (4), (5) form a *closed subsystem* of equations, which determines completely the dynamics. Putting $\Gamma = \gamma$ in the equations of the closed subsystem, we obtain the closed subsystem for the adiabatic regime, while $\Gamma \to \infty$ gives the one for the incompressible regime. The latter is also obtained by letting $\gamma \to \infty$ (formally; since $\gamma < 5/3$) in the corresponding adiabatic subsystem. Once the subsystem is solved, we can determine div $\boldsymbol{q}$ from Eq. (3) where γ now has its physical value $\gamma = c_p/c_v$. Hence for formalisms based on the subsystem only, i. e. not explicitly based on the heat flow, we have the following lemma:

LEMMA. An incompressible (non-adiabatic) formalism can be deduced from an adiabatic (compressible) formalism by letting div $\boldsymbol{v} \to 0$ *and* $\gamma \to \infty$ *while* γ div $\boldsymbol{v}$ *remains finite and provided both formalisms are not explicitly based on the heat flow.*

By formalisms can be meant e. g. energy principles for stability problems, which will be considered in the following section. *Note.* From Eq. (6) it is clear that there can be a regime which is incompressible and adiabatic at the same time. In this case, mass density and pressure are both conserved following the motion:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}\varrho}{\mathrm{d}t} = 0 \tag{11}$$

while the solutions of the system become independent of γ. Letting $\gamma \to \infty$ does not alter the present system, as is trivial for this adiabatic and incompressible regime. On the other hand, when γ *does* occur in the solution of the subsystem (1), (2), (4) and (5), we have necessarily an *adiabatic, non-incompressible* regime. Letting $\gamma \to \infty$, this adiabatic compressible regime goes over into an incompressible one, which is usually non-adiabatic. The adjectives "non-adiabatic" and "compressible" placed between brackets in the lemma and in the following have to be seen in the light of this note.

2. Comparison of the Energy Principles

In this section we consider small deviations from an equilibrium state and the connected energy principles. The perturbed equations derived from (1), (2), (3) and (5) are

$$\delta\varrho + \operatorname{div} \varrho\boldsymbol{\xi} = 0\,, \tag{12}$$

$$\varrho\ddot{\boldsymbol{\xi}} + \operatorname{grad} \delta p - \boldsymbol{f}\,\delta\varrho - \varrho\,\delta\boldsymbol{f} = 0\,, \tag{13}$$

$$\delta p + \boldsymbol{\xi}\cdot\nabla p + \gamma\, p \operatorname{div}\boldsymbol{\xi} + (\gamma - 1) \operatorname{div} \delta\boldsymbol{Q} = 0\,, \tag{14}$$

$$\delta p + \boldsymbol{\xi}\cdot\nabla p + \Gamma\, p \operatorname{div}\boldsymbol{\xi} = 0\,. \tag{15}$$

The symbols ϱ, p, Γ and $\boldsymbol{f}$ refer now to their equilibrium values. $\boldsymbol{\xi}$ is the infinitesimal displacement vector and $\boldsymbol{q}\,\delta t = \delta\boldsymbol{Q}$. To keep the discussion as general as possible, we do not detail the explicit form of the perturbed field equations. We only suppose that the perturbed fields (which occur in $\delta\boldsymbol{f}$) can be given explicitly in terms of $\boldsymbol{\xi}$ only. To obtain Eq. (15) we have assumed that the equilibrium value of Γ exists. [Note that this is not necessarily true in all physical situations. Hence Eq. (15) is not completely general, in contradistinction to Eq. (5).]

Eliminating $\delta\varrho$ and δp between Eqs. (12), (13) and (15) yields a closed system of three equations determining the three components of $\boldsymbol{\xi}$:

$$\varrho\ddot{\boldsymbol{\xi}} = \boldsymbol{F}(\boldsymbol{\xi}) \tag{16}$$

where

$$\boldsymbol{F}(\boldsymbol{\xi}) \equiv \nabla(\boldsymbol{\xi}\cdot\nabla p + \Gamma\, p \operatorname{div}\boldsymbol{\xi}) + \varrho\,\delta\boldsymbol{f}(\boldsymbol{\xi}) - \boldsymbol{f}(\boldsymbol{\xi}) \operatorname{div} \varrho\boldsymbol{\xi}\,. \tag{17}$$

For the incompressible regime we have to introduce a fourth variable

$$\alpha = \lim_{\Gamma\to\infty,\ \operatorname{div}\boldsymbol{\xi}\to 0} \Gamma\, p \operatorname{div}\boldsymbol{\xi}\,. \tag{18}$$

Now the equations

$$\varrho\ddot{\boldsymbol{\xi}} = \nabla(\boldsymbol{\xi}\cdot\nabla p + \alpha) + \varrho\,\delta\boldsymbol{f}(\boldsymbol{\xi}) - \boldsymbol{f}(\boldsymbol{\xi}) \operatorname{div} \varrho\boldsymbol{\xi}\,, \tag{19}$$

$$\operatorname{div}\boldsymbol{\xi} = 0 \tag{20}$$

make just a closed system of four equations in four unknown variables, namely α and the three components of $\boldsymbol{\xi}$.

We now proceed to the following theorem.

Theorem 1: Energy Principle for Incompressible (Non-Adiabatic) Perturbations

a. *In order to obtain the energy integral for incompressible (non-adiabatic) perturbations of a static equilibrium, it is sufficient to put* div $\boldsymbol{\xi} = 0$ *in the energy integral for adiabatic perturbations.*

b. *The energy principle technique is valid for the incompressible (non-adiabatic) regime if only those functions restricted by the constraint* $\operatorname{div}\boldsymbol{\xi}=0$ *are admitted.*

Theorem 1 owes its simplicity to the fact that no supplementary terms in $\delta\mathbf{Q}$ occur in the energy integral, which might have been expected because the incompressible perturbations are usually non-adiabatic [see Eq. (28)].

Proof a. Consider

$$\delta W(\boldsymbol{\xi},\boldsymbol{\xi}) = -\tfrac{1}{2}\int \boldsymbol{\xi}\cdot\boldsymbol{F}(\boldsymbol{\xi})\,\mathrm{d}\tau\,. \tag{21}$$

Since $\delta W(\boldsymbol{\xi},\boldsymbol{\xi})$ does not involve $\dot{\boldsymbol{\xi}}$ the operator $\boldsymbol{F}$ is self-adjoint, according to an argument due to BERNSTEIN et al.[2] and δW represents the change in potential energy. For our purpose it is sufficient to write down systematically only the terms which contain Γ. Those terms enter only in δW by means of δp. Hence, in general we have

$$\begin{aligned}\delta W &= \tfrac{1}{2}\int \boldsymbol{\xi}\cdot\operatorname{grad}\delta p\,\mathrm{d}\tau+\dots \qquad (22)\\ &= -\tfrac{1}{2}\int \delta p\operatorname{div}\boldsymbol{\xi}\,\mathrm{d}\tau+\tfrac{1}{2}\int\operatorname{div}(\delta p\,\boldsymbol{\xi})\,\mathrm{d}\tau+\dots\\ &= \tfrac{1}{2}\int \Gamma p(\operatorname{div}\boldsymbol{\xi})^2\,\mathrm{d}\tau+\tfrac{1}{2}\int\delta p\,\boldsymbol{\xi}\cdot\mathrm{d}\boldsymbol{\sigma}+\dots\,.\end{aligned}$$

The surface integral over the boundary disappears when the fluid extends to infinity. For a finite volume (surrounded by vacuum) δp appears in (22) only on the surface; δp on the surface can always be directly expressed in terms of $\boldsymbol{\xi}$ and of the vacuum quantities by means of the boundary condition. As the boundary condition is inferred from the equation of motion (13), the resulting expression for δp contains neither Γ nor γ. Hence δW is a quadratic form in the three components ξ_i

$$\delta W(\boldsymbol{\xi},\boldsymbol{\xi}) = \delta W_{\Gamma=0}(\boldsymbol{\xi},\boldsymbol{\xi}) + \tfrac{1}{2}\int \Gamma p(\operatorname{div}\boldsymbol{\xi})^2\,\mathrm{d}\tau \tag{23}$$

where $\delta W_{\Gamma=0}(\boldsymbol{\xi},\boldsymbol{\xi})$ is an expression grouping the terms not containing Γ. Since the time does not appear explicitly in $\boldsymbol{F}(\boldsymbol{\xi})$, one seeks normal mode solutions of the form $\boldsymbol{\xi}(\boldsymbol{r},t)=\boldsymbol{\xi}(\boldsymbol{r})\,e^{i\omega t}$. The corresponding eigenvalue equation is

$$-\varrho\,\omega^2\,\boldsymbol{\xi}=\boldsymbol{F}(\boldsymbol{\xi})\,. \tag{24}$$

The Euler equation of the variational principle

$$\omega^2(\boldsymbol{\xi},\boldsymbol{\xi}) = \frac{\delta W(\boldsymbol{\xi},\boldsymbol{\xi})}{K(\boldsymbol{\xi},\boldsymbol{\xi})}\,,\quad \Delta\omega^2=0 \tag{25}$$

[where $K(\boldsymbol{\xi},\boldsymbol{\xi})=\tfrac{1}{2}\int\varrho\,\xi^2\,\mathrm{d}\tau$ so that $K(\dot{\boldsymbol{\xi}},\dot{\boldsymbol{\xi}})$ is the kinetic energy] is just the eigenvalue equation (24).

The energy integral for the incompressible regime follows from Eq. (23) by letting $\Gamma\to\infty$ and $\operatorname{div}\boldsymbol{\xi}\to 0$, $\Gamma\operatorname{div}\boldsymbol{\xi}$ remaining finite. By this process, the second term on the right-hand side of Eq. (23) disappears. By merely supposing $\operatorname{div}\boldsymbol{\xi}=0$ this term disappears as well which proves theorem 1 a.

Proof b. To prove the second part of the theorem we vary $\omega^2(\boldsymbol{\xi},\boldsymbol{\xi})$ with the constraint $\operatorname{div}\boldsymbol{\xi}=0$. Using the Lagrange multiplier technique, we add the term

$$K^{-1}\int\alpha\operatorname{div}\boldsymbol{\xi}\,\mathrm{d}\tau = -K^{-1}\int\boldsymbol{\xi}\cdot\operatorname{grad}\alpha\,\mathrm{d}\tau + K^{-1}\int\alpha\,\boldsymbol{n}\cdot\boldsymbol{\xi}\,\mathrm{d}\sigma \tag{26}$$

to the right-hand side of the first equation in (25), where the multiplier α is a function of position. The effect of this term on the normal mode equation (24) is the introduction of the supplementary term $\operatorname{grad}\alpha$. Hence we obtain

$$-\omega^2\varrho\,\boldsymbol{\xi}=\boldsymbol{F}(\boldsymbol{\xi})+\operatorname{grad}\alpha \tag{27}$$

which together with the constraint $\operatorname{div}\boldsymbol{\xi}=0$ is just equivalent to the set of equations (19) – (20) for the incompressible regime. The Lagrange multiplier α plays here also the role of a fourth variable. Hence, the Lagrange multiplier α may be identified with the fourth variable defined by Eq. (18).

Remark. From Eq. (6) we have:

$$\Gamma p(\operatorname{div}\boldsymbol{\xi})^2=\gamma p(\operatorname{div}\boldsymbol{\xi})^2+(\gamma-1)\operatorname{div}\boldsymbol{\xi}\operatorname{div}\delta\mathbf{Q} \tag{28}$$

which shows that $\operatorname{div}\delta\mathbf{Q}$ does not appear in δW, neither in regime A nor in regime I. For this reason the same expression for δW is valid in both regimes. As stated before, theorem 1 owes its simplicity to this fact.

Application. We illustrate theorem 1 b by considering the energy principle for magnetodynamic systems, as exposed by CHANDRASEKHAR[7]. The energy principle for compressible, non-adiabatic and the one for incompressible, adiabatic regimes are derived by him independently from each other [his Eqs. (XIV, 30) and (XIV, 57)] and it is directly seen that his Eq. (30) follows from his Eq. (57) by setting $\operatorname{div}\boldsymbol{\xi}=0$.

Theorem 1 b can also be applied in a straightforward way e. g. to the energy principle of gravitational instability[3] and to the unified energy principle for magnetodynamic and gravitational stability[4–6]. $\delta W(\boldsymbol{\xi},\boldsymbol{\xi})$ is in this general case the sum of four terms[5].

$$\delta W=\delta W_{\mathrm{F}}+\delta W_{\mathrm{S}}+\delta W_{\mathrm{VAC}}+\delta W_{\mathrm{G}}\,. \tag{29}$$

[7] S. CHANDRASEKHAR, Hydrodynamic and Hydromagnetic Stability, Oxford University Press, London 1961.

The first three terms are in complete analogy with the paper of BERNSTEIN et al. [2]. The last term is due to self-gravitation. Thus, in the incompressible, non-adiabatic case we obtain:

$$\delta W_F = \tfrac{1}{2} \int \{ |\operatorname{curl}(\boldsymbol{\xi} \times \boldsymbol{B})|^2 - (\boldsymbol{\xi} \times \boldsymbol{j}) \cdot \operatorname{curl}(\boldsymbol{\xi} \times \boldsymbol{B}) + (\boldsymbol{\xi} \cdot \operatorname{grad} \varrho)\, \boldsymbol{\xi} \cdot \operatorname{grad}(\varphi + U) \} \, d\tau \tag{30}$$

(φ is the potential due to self-gravitation and U is an external gravitational potential). The second term on the right-hand side of Eq. (29) is a surface integral

$$\delta W_S = \tfrac{1}{2} \int (\boldsymbol{n} \cdot \boldsymbol{\xi})^2 \, [\varrho\, \boldsymbol{n} \cdot \operatorname{grad}(\varphi + U) - \langle \boldsymbol{B}\,\boldsymbol{B} \rangle : \nabla \boldsymbol{n}] \, d\sigma \tag{31}$$

where $\langle\ \rangle$ indicates here the jump at the surface of the dyadic $\boldsymbol{B}\,\boldsymbol{B}$. δW_{vac} in Eq. (29) is the magnetic energy of the vacuum:

$$\delta W_{\text{VAC}} = \tfrac{1}{2} \int |\operatorname{curl} \delta \hat{\boldsymbol{A}}|^2 \, d\tau \tag{32}$$

where $\delta\hat{\boldsymbol{A}}$ is the perturbed vector potential in the vacuum and related to the perturbation $\boldsymbol{\xi}$ by means of the boundary condition

$$\boldsymbol{n} \times \delta \hat{\boldsymbol{A}} = -(\boldsymbol{n} \cdot \boldsymbol{\xi})\, \boldsymbol{B}\,.$$

Finally, the gravitational energy is:

$$\delta W_G = -\frac{G}{2} \int \frac{(\boldsymbol{\xi} \cdot \operatorname{grad} \varrho)(\boldsymbol{\xi}' \cdot \operatorname{grad} \varrho')}{|\boldsymbol{r} - \boldsymbol{r}'|} \, d\tau \, d\tau'. \tag{33}$$

Alternative, useful forms of Eqs. (30) – (33) including the terms in div $\boldsymbol{\xi}$ are given in ref. [4] and [5] (with a different sign convention).

3. Comparison Theorems

An equilibrium state can be considered as adiabatic and incompressible at the same time ($v = 0$, $\partial/\partial t = 0$). Therefore, we can compare the stability of the same equilibrium state, now governed by adiabatic perturbations, and now by incompressible (non-adiabatic) ones.

A regime A will be called less stable than a regime I if and only if there exists a perturbation A which can grow more rapidly than any perturbation I, i. e. if all the eigenvalues of I are greater than the lowest eigenvalue of A. The notions more or equally stable are defined in an analogous way. Note that the expression "*more or less stable*" can only be defined in a consistent way by comparing growth rates and cannot be based on δW alone [8].

[8] J. G. KRÜGER and D. K. CALLEBAUT, Z. Naturforsch. **23 a**, 1357 [1968].

THEOREM 2: COMPARISON OF STABILITY (GENERAL)

An equilibrium state is more stable or at least equally stable for incompressible (non-adiabatic) perturbations than for adiabatic ones.

Proof: According to the preceding analysis, the functional $\omega^2(\boldsymbol{\xi},\boldsymbol{\xi})$ for regime A can be used in the variational principle (25) for both regimes A and I, now without and now with the constraint div $\boldsymbol{\xi} = 0$. The lowest eigenvalue ω_0^2 for regime A is given by the absolute minimum of $\omega^2(\boldsymbol{\xi},\boldsymbol{\xi})$. Any eigenvalue of regime I is given by a minimum value of the same functional $\omega^2(\boldsymbol{\xi},\boldsymbol{\xi})$ with the constraint div $\boldsymbol{\xi} = 0$ and is therefore necessarily greater than or equal to ω_0^2.

Remarks. According to theorem 2 *an incompressible regime is stable if the corresponding adiabatic regime is stable.* Theorem 2 holds *in spite of the fact* that $\delta W(\boldsymbol{\xi},\boldsymbol{\xi}) \geqq \delta W_{\Gamma=0}(\boldsymbol{\xi},\boldsymbol{\xi})$ for the same $\boldsymbol{\xi}$. Nothing can be said about the reverse unless some supplementary suppositions are introduced regarding the structure of δW, or in view of Eq. (24), regarding the structure of $\delta W_{\Gamma=0}$ (see theorems 3 and 4).

THEOREM 3: COMPARISON OF STABILITY (div $\boldsymbol{\xi}$ ARBITRARY)

If div $\boldsymbol{\xi}$ *can be chosen arbitrarily when* $\delta W_{\Gamma=0}$ *is given a fixed value, then the adiabatic and the incompressible (non-adiabatic) regimes are both stable or both unstable.*

Proof: As $\delta W_{\Gamma=0}$ has a fixed value independent of the choice of div $\boldsymbol{\xi}$, $\delta W_{\Gamma=0}$ can be bounded from below by a normalization condition which is also independent of the choice of div $\boldsymbol{\xi}$. Since $\delta W \geqq \delta W_{\Gamma=0}$, this normalization is valid for both cases A and I. With such a $\delta W_{\Gamma=0}$ and such a normalization, the second term in the right-hand side of Eq. (24) can be minimized separately for case A. Its minimum value is evidently obtained by the Euler equation

$$\operatorname{div} \boldsymbol{\xi} = 0\,. \tag{34}$$

The minimum of δW reduces to the minimum of $\delta W_{\Gamma=0}$. Taking the same normalization for case I as for case A, and as $\delta W_{\Gamma=0}$ is the same functional in both cases I and A, the minimum value of $\delta W(\boldsymbol{\xi},\boldsymbol{\xi})$ for regime A is the same value as for regime I. This means that, if δW can be made negative (respectively cannot) for case I, it can also be made negative (respectively cannot) for regime A, showing that the regimes A and I are both stable or both unstable.

Extensions. Theorem 3 can be extended to perturbations A and I which are characterized by a chosen set of parameters (see e. g. theorem 4 a). Note that theorem 3 can also be applied if div $\boldsymbol{\xi}$ can be chosen arbitrarily in an interval which contains zero.

Obviously the theorem can be extended also to cases where $\delta W_{\Gamma=0}$ can be split into a part which does not vary with div $\boldsymbol{\xi}$ and a part which has the form $\int P(\operatorname{div}\boldsymbol{\xi})^2 \, d\tau$ where $P \geqq 0$.

4. Magnetodynamics and all Cases $\delta W_{\gamma=0}$ Independent of $\xi_{\parallel}$

The theorems 2 and 3 can be elaborated for the energy principle of Bernstein and its incompressible analogue, and more generally to all cases where $\delta W_{\Gamma=0} = \delta W_{\gamma=0}$ is independent of $\xi_{\parallel}$, the component of $\boldsymbol{\xi}$ parallel to $\boldsymbol{B}$. (We mention that the following treatment can be extended to cases where one of the components of $\boldsymbol{\xi}$ is lacking in $\delta W_{\gamma=0}$.)

By judicious integration by parts the $\delta W_{\gamma=0}(\boldsymbol{\xi},\boldsymbol{\xi})$ of Bernstein et al. can be put in the form [9]

$$\begin{aligned}\delta W_{\gamma=0}(\boldsymbol{\xi},\boldsymbol{\xi}) &= \tfrac{1}{2}\int \{|\operatorname{curl}(\boldsymbol{\xi}_\perp \times \boldsymbol{B}) + \boldsymbol{n}\cdot\boldsymbol{\xi}_\perp(\boldsymbol{j}\times\boldsymbol{n})|^2 \\ &\quad - 2\,|\boldsymbol{n}\cdot\boldsymbol{\xi}_\perp|^2 \boldsymbol{j}\times\boldsymbol{n}\cdot(\boldsymbol{B}\cdot\nabla\boldsymbol{n})\}\, d\tau \qquad (35)\\ &= \delta W_{\gamma=0}(\boldsymbol{\xi}_\perp, \boldsymbol{\xi}_\perp)\end{aligned}$$

($\boldsymbol{\xi}_\perp$ = component of $\boldsymbol{\xi}$ perpendicular to $\boldsymbol{B}$, $\boldsymbol{n}$ = unit vector normal to the surface p = constant.)

From Eq. (35) it is seen that $\delta W_{\gamma=0}(\boldsymbol{\xi},\boldsymbol{\xi})$ is fixed by giving $\boldsymbol{\xi}_\perp$ alone; a normalization condition involving $\boldsymbol{\xi}_\perp$ alone is sufficient to bound $\delta W_{\gamma=0}$ from below. Now it is evident to take div $\boldsymbol{\xi}$ as a new variable instead of $\xi_{\parallel}$. In order to make the elaboration of theorem 3 possible, it is necessary to know whether the *new variable* div $\boldsymbol{\xi}$ *can be chosen arbitrarily due to the free choice of* $\xi_{\parallel}$. Yet owing to the existence of closed field lines e. g. this is *not* always the case. Indeed, the transformation between the old variable $\xi_{\parallel}$ and the new one, div $\boldsymbol{\xi}$, is given by the relation

$$\operatorname{div}\boldsymbol{\xi} = \operatorname{div}\boldsymbol{\xi}_\perp + \operatorname{div}\boldsymbol{\xi}_{\parallel} \qquad (36)$$

where div $\boldsymbol{\xi}_\perp$ is supposed to be known. If $\xi_{\parallel}$ (and $\boldsymbol{\xi}_\perp$) is given, div $\boldsymbol{\xi}$ is unambiguously determined. Conversely, if div $\boldsymbol{\xi}$ (and $\boldsymbol{\xi}_\perp$) is gven, $\xi_{\parallel}$ is determined by the "magnetic differential equation"

$$\boldsymbol{B}\cdot\operatorname{grad}(\xi_{\parallel}/B) = s \qquad (37)$$

where

$$s \equiv \operatorname{div}\boldsymbol{\xi} - \operatorname{div}\boldsymbol{\xi}_\perp \qquad (38)$$

is known. The solution of Eq. (37) is

$$\xi_{\parallel}(P) = B\int (s/B)\, dl + \xi_{\parallel}(P_0) \qquad (39)$$

where the integration is taken along a field line from an arbitrary point P_0 to the point P. Two restrictive conditions on s will be discussed in sections 4.1 and 4.2.

4.1. *Restrictive Condition* $\langle s \rangle_{\text{tube}} = 0$

Along a closed field line, the requirement that the continuous function $\xi_{\parallel}$ is single-valued yields the condition

$$\oint s B^{-1}\, dl = 0. \qquad (40)$$

NEWCOMB [10] proved that this equation is a necessary and sufficient condition in order that the magnetic differential equation (37) has a continuous and single-valued solution. Denoting by $\langle X \rangle_{\text{tube}}$ the average value of any quantity X over a closed flux tube, we have

$$\langle s \rangle_{\text{tube}} = \frac{\int_{\text{tube}} s\, d\tau}{\int_{\text{tube}} d\tau} = \frac{\oint s B^{-1}\, dl}{\oint B^{-1}\, dl}. \qquad (41)$$

Taking the average of Eq. (37) over a closed flux tube, we obtain condition (40) which may be written equivalently

$$\langle s \rangle_{\text{tube}} = 0. \qquad (42)$$

Eq. (42) restricts, in view of Eq. (38) the free choice of div $\boldsymbol{\xi}$.

Besides configurations characterized by magnetic surfaces which are torusses, we also consider configurations characterized by infinitely long tubes e. g. cylinders as in the illustration below. The infinitely long tube has only a physical significance if it is thought of as the limit of a toroidal configuration with a tube length very much greater than its diameter. The limiting process implies that we have here also the condition (42) where the average is now taken along the whole open field line. If the tube and the perturbations are periodic along the tube axis, the average can be taken over one common period.

Classification: $\langle s \rangle_{\text{tube}} \equiv 0$ or $\langle s \rangle_{\text{tube}} \not\equiv 0$.

[9] See Eq. (4.5) of the paper of BERNSTEIN et al. [2]

[10] W. NEWCOMB, Phys. Fluids **2**, 362 [1959].

Two cases may occur with respect to the condition $\langle s \rangle_{\text{tube}} = 0$.

a. $\langle s \rangle_{\text{tube}} \equiv 0$. The condition $\langle s \rangle_{\text{tube}} = 0$ is identically fulfilled with respect to the variable div $\boldsymbol{\xi}$ and $\boldsymbol{\xi}_{\perp}$ for certain specific perturbations.

b. $\langle s \rangle_{\text{tube}} \not\equiv 0$. This is the negation of case a.

The expression "identically fulfilled" has to be understood in a definite manner. One way to define it is as follows. Expand the quantity X be means of the orthogonal functions ψ_i

$$X = \sum_i X_i \psi_i \tag{43}$$

where X_i are constant coefficients and where i stands for any number of parameters $k, m, \ldots$. In this way, the independent functions div $\boldsymbol{\xi}$ and div $\boldsymbol{\xi}_{\perp}$ are parametrized. Their coefficients $(\text{div}\,\boldsymbol{\xi})_i$ and $(\text{div}\,\boldsymbol{\xi}_{\perp})_i$ have to be considered as independent. The condition $\langle s \rangle_{\text{tube}} = 0$ leads to

$$\sum_i s_i \langle \psi_i \rangle_{\text{tube}} = 0 \tag{44}$$

which yields a relation between all these independent coefficients. We say that $\langle s \rangle_{\text{tube}} = 0$ is identically fulfilled with respect to the new variables div $\boldsymbol{\xi}$ and $\boldsymbol{\xi}_{\perp}$ if Eq. (40) is identically fulfilled with respect to the independent coefficients $(\text{div}\,\boldsymbol{\xi})_i$ and $(\text{div}\,\boldsymbol{\xi}_{\perp})_i$. As

$$s_i = (\text{div}\,\boldsymbol{\xi})_i - (\text{div}\,\boldsymbol{\xi}_{\perp})_i \tag{45}$$

this is equivalent by requiring that Eq. (40) is identically fulfilled for arbitrary s_i. Hence, to say that $\langle s \rangle_{\text{tube}} \equiv 0$ on a flux tube labeled by parameters α and β requires

$$\langle \psi_i \rangle_{\text{tube}} = 0 \tag{46}$$

on this flux tube. In general Eq. (46) may be satisfied for a set S of parameters i, α, β. Furthermore we choose the functions ψ_i so that each of the perturbations $\boldsymbol{\xi}_i$ corresponding to a coefficient s_i form a set of independent orthogonal functions. In this way each perturbation i can be considered separately.

For the set S of parameters i, α, β for which Eq. (46) is valid, $(\text{div}\,\boldsymbol{\xi})_i$ can be chosen arbitrarily whatever the value of $\boldsymbol{\xi}_{\perp}$ may be. Hence div $\boldsymbol{\xi}$ can be chosen arbitrarily for any $\boldsymbol{\xi}_{\perp}$ for the specific perturbations characterized by the set S.

When $\langle s \rangle_{\text{tube}} \not\equiv 0$, Eq. (46) is not satisfied for a set of parameters, i, α, β complementary to the set S. Clearly $\langle s \rangle_{\text{tube}} = 0$ or

$$\langle \text{div}\,\boldsymbol{\xi} \rangle_{\text{tube}} = \langle \text{div}\,\boldsymbol{\xi}_{\perp} \rangle_{\text{tube}} \tag{47}$$

yields then a condition between the coefficients $(\text{div}\,\boldsymbol{\xi})_i$ and $(\text{div}\,\boldsymbol{\xi}_{\perp})_i$ for the parameters considered. We shall say shortly: for the specific perturbations considered $\langle s \rangle_{\text{tube}}$ is *not identically* fulfilled with respect to div $\boldsymbol{\xi}$ and div $\boldsymbol{\xi}_{\perp}$. The condition $\langle s \rangle_{\text{tube}} = 0$ does imply a restriction on the free choice of div $\boldsymbol{\xi}$ when $\boldsymbol{\xi}_{\perp}$ is supposed to be known. The new set of variables div $\boldsymbol{\xi}$ and $\boldsymbol{\xi}_{\perp}$ *cannot* be considered as independent variables for the set of parameters considered.

Example of the Linear Pinch

To illustrate the occurrence of the two cases a and b we consider the stability theory of a diffuse linear pinch [11], based on a magnetic field

$$\boldsymbol{B} = [0, B_{\varphi}(r), B_z(r)] \tag{48}$$

in cylindrical coordinates, the field components depending only on the distance r to the cylinder axis. Each perturbed quantity X can be Fourier-analyzed in φ and z. It is sufficient to consider a single Fourier component

$$X = X_{k,m} \exp\{i(k z + m \varphi)\}\,. \tag{49}$$

Along a field line, every perturbed quantity X can be written as

$$X = X_{k,m}(r, \beta) \exp\{i[(k B_z + m B_{\varphi}/r)\, l/B]\} \tag{50}$$

where l is the length along the field line and where β labels the field line on the pressure shell.

a. $\langle s \rangle_{\text{tube}} \equiv 0$. Taking the average over one period of the field line, the left-hand side of Eq. (50) yields zero for all values of $s_{k,m}$ provided that

$$k\, r\, B_z + m\, B_{\varphi} \neq 0\,. \tag{51}$$

The k and m values satisfying Eq. (51) characterize the set S of the parameters k and m for which $\langle s \rangle_{\text{tube}} = 0$ is identically fulfilled for arbitrary $s_{k,m}$. From our general considerations it follows that div $\boldsymbol{\xi}$ can be chosen arbitrarily due to the free choice of $\xi_{\parallel}$. This can be seen directly from Eq. (36) which becomes on account of Eq. (49) a linear algebraic equation

$$(\text{div}\,\boldsymbol{\xi})_{k,m} = (\text{div}\,\boldsymbol{\xi}_{\perp})_{k,m} + (k B_z + m B_{\varphi}/r)\, (\xi_{\parallel})_{k,m}/B \tag{52}$$

[11] W. NEWCOMB, Ann. Phys. New York **10**, 232 [1960].

demonstrating that there is a one-to-one correspondence between $(\operatorname{div} \boldsymbol{\xi})_{k,m}$ and $(\xi_{\|})_{k,m}$.

b. $\langle s \rangle_{\text{tube}} \not\equiv 0$. $\langle s \rangle_{\text{tube}}$ is not identically zero for all $s_{k,m}$ if there are k and m values related by

$$k\, r\, B_z + m\, B_\varphi = 0 . \tag{53}$$

This equation defines a set of parameters k, m complementary to S. Case b can only occur either when $k = m = 0$ or when the pitch $\mu/2\pi$ is constant; μ stands for

$$\mu = B_\varphi / (r\, B_z) . \tag{54}$$

For the set of parameters k and m defined by Eq. (53) $\langle s \rangle_{\text{tube}} = 0$ yields a restrictive condition on the new variable $(\operatorname{div} \boldsymbol{\xi})_{-\mu m, m}$

$$(\operatorname{div} \boldsymbol{\xi})_{-\mu m, m} - (\operatorname{div} \boldsymbol{\xi}_\perp)_{-\mu m, m} = 0 . \tag{55}$$

Hence $\operatorname{div} \boldsymbol{\xi}$ *cannot* be chosen arbitrarily. This again can be seen directly from Eq. (52) which allows *no* one-to-one correspondence between $(\operatorname{div} \boldsymbol{\xi})_{-\mu m, m}$ and $(\boldsymbol{\xi}_{\|})_{-\mu m, m}$.

Minimization of the Energy Integral

We intend to minimize δW with respect to the new variable $\operatorname{div} \boldsymbol{\xi}$ taking a normalization condition not containing $\xi_{\|}$. We shall first consider the adiabatic regime with $\Gamma = \gamma$. It is sufficient to obtain the minimum of the integral

$$J = \tfrac{1}{2} \int \gamma\, p (\operatorname{div} \boldsymbol{\xi})^2 \, d\tau . \tag{56}$$

In case a, where $\operatorname{div} \boldsymbol{\xi}$ may be taken arbitrarily, the minimum of this integral is obviously zero (compare with theorem 3) and $\operatorname{div} \boldsymbol{\xi} = 0$ is the Euler equation. In case b, where $\operatorname{div} \boldsymbol{\xi}$ cannot be taken arbitrarily, we may have a minimum different from zero which we shall determine now.

Following the usual Lagrange multiplier technique we take into account the conditions (40) on every closed field line by adding the following term to the integral (56)

$$\int \alpha [\oint B^{-1} \operatorname{div} \boldsymbol{\xi}_\perp \, dl - \oint B^{-1} \operatorname{div} \boldsymbol{\xi} \, dl] \, d\psi . \tag{57}$$

α is an arbitrary function of position which is constant on the field lines $(\boldsymbol{B} \cdot \operatorname{grad} \alpha = 0)$. The integration is carried out with respect to the flux ψ. The expression (57) is nothing but the volume integral

$$\int \alpha (\operatorname{div} \boldsymbol{\xi}_\perp - \operatorname{div} \boldsymbol{\xi}) \, d\tau . \tag{58}$$

From Eqs. (56) and (58) the Euler equation with respect to $\operatorname{div} \boldsymbol{\xi}$ is

$$\gamma\, p \operatorname{div} \boldsymbol{\xi} - \alpha = 0 . \tag{59}$$

For case a we can put $\alpha = 0$ and recover the simple Euler equation $\operatorname{div} \boldsymbol{\xi} = 0$. Eq. (59) shows that the extremizing value of $\operatorname{div} \boldsymbol{\xi}$ or $dp = \gamma\, p \operatorname{div} \boldsymbol{\xi}$ is constant on the field lines. This fact can be derived generally (without the use of closed field lines) by variation of the integral J with respect to $\xi_{\|}$. Indeed, transforming Eq. (56) into

$$J = -\tfrac{1}{2} \int \boldsymbol{\xi} \cdot \nabla (\gamma\, p \operatorname{div} \boldsymbol{\xi}) \, d\tau + \tfrac{1}{2} \int \gamma\, p (\operatorname{div} \boldsymbol{\xi})\, \boldsymbol{n} \cdot \boldsymbol{\xi} \, d\sigma \tag{60}$$

one obtains upon variation the Euler equation

$$\boldsymbol{B} \cdot \operatorname{grad} (\gamma\, p \operatorname{div} \boldsymbol{\xi}) = 0 \tag{61}$$

from which Eq. (59) follows immediately. From $\langle s \rangle_{\text{tube}} = 0$ the constant value of $\operatorname{div} \boldsymbol{\xi}$ on each field line is

$$\operatorname{div} \boldsymbol{\xi} = \langle \operatorname{div} \boldsymbol{\xi}_\perp \rangle_{\text{tube}} . \tag{62}$$

Comparison Between Perturbations of an Infinitely Long Tube

It is worthwhile to write out the Euler equation for the three components of $\boldsymbol{\xi}$

$$\operatorname{grad} dp(\boldsymbol{\xi}) + \boldsymbol{F}_\perp (\boldsymbol{\xi}) = 0 . \tag{63}$$

The gradient comes from the integral J, [see Eq. (56)], the second term results from the term $\delta W_{\gamma=0}$ and the normalization. In case a the extremizing dp is zero, in case b it is a function which is constant on the field lines. This has interesting consequences for the infinitely long tube. In this case we suppose (as can be shown to hold for most of such configurations) that there are values of the parameters belonging to case a which can be chosen infinitely close to definite values of the parameters belonging to case b so that the conditions characterizing the case a become *infinitely close* to the conditions characterizing case b. However, the minimizing $\boldsymbol{\xi}_a$ belonging to case a does not necessarily converge to the minimizing $\boldsymbol{\xi}_b$ belonging to case b because $\boldsymbol{\xi}_a$ and $\boldsymbol{\xi}_b$ are solutions of Euler equations which have a different structure in each case. Hence, the minimum δW_a belonging to case a *does not necessarily converge* to the minimum δW_b belonging to case b if the values of the parameters belonging to case a converge to the values of the parameters of case b. If the limit of the minimum δW_a exists, it will be in general different from the minimum δW_b: in general there will be a *discontinuity*. A question which arises naturally is: which one of the energy integrals has the lowest minimum?

The minimum δW_b has been obtained by taking into account the constraint $\langle s \rangle_{tube} = 0$, while the limit of the minimum follows from a minimization without constraint. As we compare these minima for the same values of parameters (and as the same normalization is taken) we have

$$\lim \delta W_a \leqq \delta W_b \tag{64}$$

for the parameters considered. The importance of this equation lies in the fact that it is *only* necessary to look for the case a if one is only interested in stability analyses.

Comparison Between Adiabatic and Incompressible Regimes

Since the incompressible (non-adiabatic) regime can be derived from the adiabatic (compressible) one according to the basic lemma we can associate with each perturbation belonging to case a or b of the adiabatic regime a corresponding perturbation of the incompressible regime. Hence, the same division in two cases a and b applies to the incompressible regime as well. The treatment of the infinitely long tube can be taken over completely. The two cases a and b are characterized by the form of dp (= limit of $\gamma\, p$ div $\boldsymbol{\xi}$) which again is zero in case a and constant on the field lines in case b. Eq. (64) remains also valid for the incompressible regime. The correspondence between the perturbations of the regime A and I enables us to compare the stability of A and I for cases a and b.

Case a. ($\langle s \rangle_{tube} \equiv 0$.) The suppositions required for the application of theorem 3 are satisfied for the values of the parameters under consideration. Hence: the regimes A and I are *both stable or both unstable* for the parameters considered. ($\delta W_A = \delta W_I$ for the minimized energy integrals.)

Case b. ($\langle s \rangle_{tube} \not\equiv 0$.) The suppositions required for the application of theorem 3 are not satisfied for the parameters considered. Then, from theorem 2 we have still the less strong statement $\delta W_A \leqq \delta W_I$ (minimized energy integrals) for the values of the parameters under consideration.

THEOREM 4.
COMPARISON OF STABILITY FOR MAGNETOSTATIC EQUILIBRIA WITH $\delta W_{\gamma=0}$ INDEPENDENT OF $\xi_{\parallel}$

a. *If div $\boldsymbol{\xi}$ can be chosen arbitrarily (i. e. if $\langle s \rangle_{tube} \equiv 0$) for certain values of the parameters characterizing the perturbations, then the incompressible (non-adiabatic) and the adiabatic (compressible) regimes are both stable or both unstable for the values of the parameters considered.*

b. *If div $\boldsymbol{\xi}$ cannot be chosen arbitrarily (i. e. if $\langle s \rangle_{tube} \not\equiv 0$) for certain values of the parameters characterizing the perturbations, while, however, these values can be approached infinitely close by the values of parameters characterizing perturbations for which* div $\boldsymbol{\xi}$ *can be chosen arbitrarily ($\langle s \rangle_{tube} \equiv 0$), then the adiabatic and the incompressible regimes are both stable or both unstable for the values of the parameters considered.*

COROLLARY

If all the values of the parameter belong either to part a or to part b of theorem 4, then

a. *the adiabatic and the incompressible regimes are both stable or both unstable* (irrespective of the parameters considered);

b. *it is sufficient to consider only perturbations for which $\langle s \rangle_{tube} \equiv 0$ and for which* div $\boldsymbol{\xi} = 0$ *in order to decide whether the equilibrium is stable or not (in both cases A and I).*

It is clear that the corollary allows a considerable simplification in testing the system for stability. The suppositions required for the application of the corollary are satisfied for the example of the linear pinch. This property is related to the infinite extension of the plasma cylinder (see section 4.3). It can be made plausible that the same property still holds for infinitely long plasma tubes of any shape. Hence, *for infinitely long tubes the regimes A and I are in general both stable or both unstable* and in order to decide whether the equlibrium is stable or not *it is sufficient to consider only perturbations for which $\langle s \rangle_{tube} \equiv 0$ and for which* div $\boldsymbol{\xi} = 0$.

Case	Constraint on div $\boldsymbol{\xi}$	Constraint on minimization	Comparison of neighbouring states	Comparison of regimes A and I	Example: linear pinch.
a	$\langle s \rangle_{tube} \equiv 0$	no	$\lim \delta W_a \leqq \delta W_b$	$\delta W_A = \delta W_I$	$k + m\,\mu \neq 0$
b	$\langle s \rangle_{tube} \not\equiv 0$	yes		$\delta W_A \leqq \delta W_I$	$k + m\,\mu = 0$

Table 1. Properties of perturbations. This table is the summary of the results of section 4.1, which analyses the constraint $\langle s \rangle_{tube} = 0$. $s = \text{div}\,\boldsymbol{\xi} - \text{div}\,\boldsymbol{\xi}_{\perp}$, δW stands for the minimized energy integral and μ is 2π divided by the pitch.

The results of the whole section 4.1 are summarized in Table 1, together with the application to the linear pinch.

4.2. *Restrictive Condition* $\langle s \rangle_{\text{shell}} = 0$

The preceding case b may be further analysed in an entirely analogous way as in the preceding section 4.1. Denoting by $\langle X \rangle_{\text{shell}}$ the average value of any quantity X over a shell between two neighbouring pressure surfaces, we have

$$\langle s \rangle_{\text{shell}} = \frac{\int\limits_{\text{shell}} s \, d\tau}{\int\limits_{\text{shell}} d\tau} = \frac{\int s \, |\nabla p|^{-1} \, d\sigma}{\int |\nabla p|^{-1} \, d\sigma} . \qquad (65)$$

Taking the average of Eq. (37) over a pressure shell we obtain the restrictive condition

$$\langle s \rangle_{\text{shell}} = 0 . \qquad (66)$$

As is explained in detail by NEWCOMB [10], $\langle s \rangle_{\text{tube}} = 0$ implies $\langle s \rangle_{\text{shell}} = 0$ on every pressure shell, even on ergodic surfaces. Hence $\langle s \rangle_{\text{tube}} \equiv 0$ implies $\langle s \rangle_{\text{shell}} \equiv 0$ ("identically fulfilled" has to be understood in an analogous way as explained in section 4.1.).

Classification: $\langle s \rangle_{\text{shell}} \equiv 0$ or $\langle s \rangle_{\text{shell}} \not\equiv 0$.

When $\langle s \rangle_{\text{tube}} \not\equiv 0$, two cases a and b may occur, similar to the cases a and b of section 4.1.

a. $\underline{\langle s \rangle_{\text{shell}} \equiv 0}$. The condition $\langle s \rangle_{\text{shell}} = 0$ is identically fulfilled with respect to the variables div $\boldsymbol{\xi}_\perp$ and $\boldsymbol{\xi}_p$ for certain specific perturbations. $\boldsymbol{\xi}_p$ is the component of $\boldsymbol{\xi}$ perpendicular to the pressure (or magnetic) surfaces. Following an analogous argument as in section 4.1, this means that

$$\sum_i s_i \langle \psi_i \rangle_{\text{shell}} = 0 \qquad (67)$$

for arbitrary s_i, leading to

$$\langle \psi_i \rangle_{\text{shell}} = 0 \qquad (68)$$

Eq. (68) may be satisfied for a set of labels i, α, β characterizing perturbations for which $\langle s \rangle_{\text{shell}} \equiv 0$.

b. $\underline{\langle s \rangle_{\text{shell}} \not\equiv 0}$. Then Eq. (68) is not satisfied for a set of labels i, α, β complementary to the set considered under a. $\langle s \rangle_{\text{shell}} = 0$ leads to

$$\langle \text{div}\, \boldsymbol{\xi} \rangle_{\text{shell}} = \langle \text{div}\, \boldsymbol{\xi}_\perp \rangle_{\text{shell}} = \langle \text{div}\, \boldsymbol{\xi}_p \rangle_{\text{shell}} . \qquad (69)$$

The last equality does imply a restriction on the free choice of $\langle \text{div} \boldsymbol{\xi}_\perp \rangle_{\text{tube}}$ when $\boldsymbol{\xi}_p$ is supposed to be known. In order to compare the conditions (42) and (66) we remark that $\langle s \rangle_{\text{tube}} = 0$ on every tube p, β is equivalent with the set of equations

$$\langle f_n(p, \beta) \, s \rangle_{\text{shell}} = 0 , \qquad n = 0, 1, 2, \ldots , \qquad (70)$$

where $f_n(p, \beta)$ are a complete set of independent functions which are constant on a flux tube (p, β). Each index n characterizes a perturbation for which $\langle s \rangle_{\text{tube}} \not\equiv 0$. Taking f_0 constant, the perturbation characterized by $\langle s \rangle_{\text{shell}} = 0$ is just the one characterized by the index $n = 0$. Hence, for perturbations for which $\langle s \rangle_{\text{shell}} \not\equiv 0$, $\langle s \rangle_{\text{shell}} = 0$ *is the only constraint.*

Example of the Linear Pinch

We illustrate the occurrence of the two cases a and b by considering again the diffuse linear pinch. The surfaces of constant pressure are circular cylinders $r = \text{constant}$. The average over these surfaces may be taken over one period in the z-direction.

a. $\underline{\langle s \rangle_{\text{shell}} \equiv 0}$. In view of Eq. (49) $\langle s \rangle_{\text{shell}}$ is zero for arbitrary values of $s_{k,m}$ provided that

$$k \quad \text{or} \quad m \neq 0 . \qquad (71)$$

This condition is obviously fulfilled if

$$k \, r \, B_z + m \, B_\varphi \neq 0$$

as it should be. Eq. (71) defines a subset of parameters k, m belonging to the set $k \, r \, B_z + m \, B_\varphi = 0$.

b. $\underline{\langle s \rangle_{\text{shell}} \not\equiv 0}$. $\langle s \rangle_{\text{shell}}$ is not identically zero unless

$$k = m = 0 \qquad (72)$$

which defines the set complementary to the set of a. The restrictive condition on div$\boldsymbol{\xi}_\perp$ from requiring $\langle s \rangle_{\text{shell}} = 0$ is

$$(\text{div}\, \boldsymbol{\xi}_\perp)_{0,0} = \left(\frac{d\xi_r}{dr} + \frac{\xi_r}{r} \right)_{0,0} \qquad (73)$$

which is contained in Eq. (55) as it should be. Note that the requirement $\langle s \rangle_{\text{shell}} = 0$ is equivalent to the requirement $\langle s \rangle_{\text{tube}} = 0$ for $k = m = 0$.

Minimization of the Energy Integral

The minimized form of δW with respect to $\xi_\parallel$ (or with respect to the new variable div $\boldsymbol{\xi}$) is

$$\delta W = \int \gamma \, p \langle \text{div}\, \boldsymbol{\xi}_\perp \rangle_{\text{tube}}^2 \, d\tau + \delta W_{\gamma=0}(\boldsymbol{\xi}_\perp, \boldsymbol{\xi}_\perp) \qquad (74)$$

which contains indeed only $\boldsymbol{\xi}_\perp$, not $\xi_\parallel$. As $\langle s \rangle_{\text{shell}} = 0$ is the only constraint for the perturbations considered, it is sufficient to introduce a Lagrange multiplier α on every pressure shell. A similar derivation as for case b of section 4.1 gives us the Euler equation (59) where α (or dp) is now a function of p only. Hence, div $\boldsymbol{\xi}$ is not only constant on every flux tube but even on every pressure shell and from

Case	Constraint on $\langle \operatorname{div} \boldsymbol{\xi} \rangle_{\text{tube}}$	Constraint on minimization	Comparison of neighbouring states	Example: linear pinch
a	$\langle s \rangle_{\text{shell}} \equiv 0$	no	$\lim \delta W_{\text{a}} \leqq \delta W_{\text{b}}$	k or $m \neq 0$
b	$\langle s \rangle_{\text{shell}} \not\equiv 0$	yes		$k = m = 0$

Table 2. Properties of perturbations for which $\langle s \rangle_{\text{tube}} \not\equiv 0$. This table is the summary of the results of section 4.2 which analyses the constraint $\langle s \rangle_{\text{shell}} = 0$. $s = \operatorname{div} \boldsymbol{\xi} - \operatorname{div} \boldsymbol{\xi}_\perp$, δW stands for the minimized energy integral.

$\langle s \rangle_{\text{shell}} = 0$ we obtain

$$\operatorname{div} \boldsymbol{\xi} = \langle \operatorname{div} \boldsymbol{\xi}_\perp \rangle_{\text{tube}} = \langle \operatorname{div} \boldsymbol{\xi}_{\text{p}} \rangle_{\text{shell}} . \quad (75)$$

The last equality implies an *additional* constraint on $\boldsymbol{\xi}_\perp$. The limit of the minimum δW_{b} is obtained by minimizing δW from Eq. (74) further with respect to $\boldsymbol{\xi}_\perp$ by taking into account the constraint (75), while the minimum δW_{a} is obtained without constraint. Hence

$$\lim \delta W_{\text{a}} \leqq \delta W_{\text{b}} \quad (76)$$

for the values of the parameters under consideration. Repeating an analogous reasoning as in section 4.1 it is clear that Eq. (76) is also valid for the incompressible regime. *If one is only interested in stability it is only necessary to consider case a.*

The results of this section, together with the example of the linear pinch, are summarized in Table 2.

4.3. Illustration: The Linear Pinch

We illustrate the preceding analysis of sections 4.1 and 4.2 explicitly for a particular equilibrium configuration: the diffuse linear pinch with a magnetic field given by Eq. (48).

Classification of the Perturbations

As case 1 b of section 4.1 is divided in the two cases 2 a and 2 b in section 4.2 we have to consider three distinct cases:

$$\begin{array}{ll} 1\,\text{a}: & k\,r\,B_z + m\,B_\varphi \neq 0 \\ 1\,\text{b}: & \left\{ \begin{array}{ll} 2\,\text{a}: & k\,r\,B_z + m\,B_\varphi = 0 \text{ but } \; k \neq 0 \text{ or } m \neq 0 \\ 2\,\text{b}: & k = 0 \text{ and } m = 0 \end{array} \right. \end{array} \quad (77)$$

The case 2 a can only occur for constant pitch. Cases 1 a and 2 b occur whether the pitch is constant or not. Cases 1 a and 2 b were treated by Newcomb [11]. Case 2 a is considered in a later article by Tayler [12].

Minimization of the Energy Integral

After minimization with respect to ξ_φ and ξ_z, δW can be brought in the form

[12] R. J. Tayler, Plasma Phys., J. Nucl. Energy Part C **3**, 266 [1961]. In this article an external potential is taken into account.

$$\delta W(\xi_r, \xi_r) = \tfrac{1}{2} \pi \int (P\,\xi_r'^2 + 2\,Q\,\xi_r'\,\xi_r/r + R\,\xi_r^2/r^2)\; r\,\mathrm{d}r \quad (78)$$

where surface terms are supposed to vanish.

To arrive at this form, we need two Euler equations in case 1 a. For one of these Euler equations one can take $\operatorname{div} \boldsymbol{\xi} = 0$. This agrees with a result of Newcomb [11] [his Eq. (13 a)]. In case 2 a, as there is one constraint, there remains only one Euler equation with respect to ξ_φ and ξ_z which reads (see Tayler [12]):

$$\operatorname{div} \boldsymbol{\xi} = 2\,B_\varphi^2\,\xi_r/(B^2 + \gamma\,p)\,r . \quad (79)$$

Eq. (79) shows that $\operatorname{div} \boldsymbol{\xi}$ is in general non-zero, in agreement with the general treatment. In case 2 b, since there are two constraints, there remains *no* Euler equation with respect to ξ_φ and ξ_z. $\operatorname{div} \boldsymbol{\xi}$ is now equal to $\xi_r' + \xi_r/r$. Except for the case $\xi_r = 0$, we see here also that $\operatorname{div} \boldsymbol{\xi}$ is non-zero for physical perturbations.

As a consequence of the difference in Euler equations, the coefficients P, Q and R of Eq. (78) have a different structure in each of the three cases; they are listed in Table 3. They are chosen so that the derivatives of the equilibrium quantities do not occur in them. This has the advantage that the sign of each term occuring in P, Q and R is well-known.

Comparison Between the Three Cases

Due to the fact that k varies continuously (and this relies on the infinite extension of the cylinder), the conditions characterizing case 1 a (respectively 2 a) may approach infinitely close to the conditions characterizing cases 1 b (respectively 2 b). Hence we can compare one case with another one taken for the corresponding *limiting* condition.

Although m takes only integer values physically the following inequalities are derived for all real values of m. This may be of interest for constructing other useful inequalities [13].

[13] The same remark may apply to the k-values of a torus. In a torus k takes only discrete values. However, from the stability for "mathematical" k-values in the vicinity of the physical ones, it may be possible to deduce stability for the latter ones.

	P	Q	R
1 a $k r B_z + m B_\varphi \neq 0$	$\frac{(k r B_z + m B_\varphi)^2}{k^2 r^2 + m^2}$	$\frac{k^2 r^2 B^2}{k^2 r^2 + m^2}$	$\frac{(k r B_z - m B_\varphi)^2}{k^2 r^2 + m^2} + (k r B_z + m B_\varphi)^2 - 2 B_\varphi^2$
2 a $k r B_z + m B_\varphi = 0$ k or $m \neq 0$	0	B_φ^2	$2 B_\varphi^2 - \frac{4 B_\varphi^4}{B^2 + \gamma p}$
2 b $k = m = 0$	$B^2 + \gamma p$	$B_z^2 + \gamma p$	$B_z^2 - B_\varphi^2 + \gamma p$

Table 3. Different forms of δW for the linear pinch. The coefficients appearing in Eq. (78) are listed for the three cases distinguished by Eqs. (77). The minimized energy integral is $\delta W = \frac{1}{2} \pi \int (P \xi_r'^2 + 2 Q \xi_r' \xi_r / r + R \xi_r^2 / r^2) r \, dr$.

Discontinuity Between Cases 1 a $(k + m \mu \to 0)$ and 2 a $(k + m \mu = 0)$

The comparison deals with magnetic fields of constant pitch. From Table 3 we obtain

$$\delta W_{2a} = \lim \delta W_{1a} + 2 \pi \int \frac{B_\varphi^2}{r} \left(\frac{B_\varphi^2}{B^2 + \gamma p} + \frac{k^2 r^2}{k^2 r^2 + m^2} \right) \xi_r^2 \, dr \geqq \lim \delta W_{1a} . \tag{80}$$

Hence a linear pinch of constant pitch[14] $2 \pi / \mu$ *is stable for* $k = -m \mu$ *if it is stable for* $k \to -m \mu$.

Discontinuity Between Cases 1 a $(k \to 0,\ m \to 0)$ and 2 b $(k = 0,\ m = 0)$

Putting $q = \lim k/m$ as $k \to 0$ and $m \to 0$, we obtain from Table 3

$$\delta W_{2b} = \lim \delta W_{1a} + \frac{\pi}{2} \int \left\{ \gamma p (\xi_r' + \xi_r / r)^2 + \frac{[q r B_\varphi (\xi_r' - \xi_r / r) - B_z (\xi_r' + \xi_r / r)]^2}{q^2 r^2 + 1} \right\} r \, dr \geqq \lim \delta W_{1a} . \tag{81}$$

Hence a linear pinch is stable for $k = 0$, $m = 0$ *if it is stable for* $k \to 0$, $m \to 0$. For integer m we have to put first $m = 0$ and then let $k \to 0$, which corresponds to $q = \infty$. Then Eq. (81) reduces to an inequality derived by NEWCOMB[11] [his Eq. (22)]. We point out, however, that Eq. (81) is not valid for fields of constant pitch $2 \pi / \mu$ for which $q + \mu = 0$, which will be treated in the following paragraph.

Discontinuity Between Cases 2 a $(k + m \mu = 0,\ k \to 0,\ m \to 0)$ and 2 b $(k = 0,\ m = 0)$

The comparison deals with fields of constant pitch. From Table 3 we obtain

$$\delta W_{2b} = \lim \delta W_{2a} + \frac{\pi}{2} \int (B^2 + \gamma p) \left[\xi_r' + \left(1 - \frac{2 B_\varphi^2}{B^2 + \gamma p} \right) \frac{\xi_r}{r} \right]^2 r \, dr \geqq \lim \delta W_{2a} . \tag{82}$$

Hence a linear pinch of constant pitch is stable for $k = 0$, $m = 0$ *if it is stable for* $k + m \mu = 0$, $k \to 0$, $m \to 0$. For integer m, where we have to put $m = 0$ first and then let $k \to 0$, the comparison deals only with purely transverse fields ($B_z = 0$, $B_\varphi \neq 0$).

The three equations (80) – (82) which hold for every trial function ξ_r will hold as well for the completely minimized energy integrals. This confirms and illustrates the general treatment. The three inequalities (80) – (82) allow considerable simplification in testing the pinch for stability. If we wish to find out whether a specified pinch is stable for all values of k and m, *we need only to examine case 1 a* $(k r B_z + m B_\varphi \neq 0)$. This holds as well for variable as for constant pitch. As $\delta W(\xi_r, \xi_r)$ is independent of γ in case 1 a, the stability criteria will be independent of γ.

Analysis of the Incompressible Regime

The conditions (77) define also the three cases in the incompressible regime. The Eqs. (80) – (82) remain valid if $\gamma \to \infty$, Eqs. (81) and (82) in a trivial way. Indeed, if $k = m = 0$ or $k \to 0$, $m \to 0$, the condition div $\boldsymbol{\xi} = 0$ becomes $(\xi_r' + \xi_r / r)_{0,0} = 0$,

[14] A field of constant pitch surrounding the central plasma column was studied experimentally by S. A. COLGATE, J. P. FERGUSON, H. P. FURTH, and R. E. WRIGHT, Proc. 2nd UN Intern. Conf. PUAE **32**, 140 [1958] in the case of a linear pinch and more recently by the group working in Jutphaas (the Netherlands) in the case of a toroidal pinch (Novosibirsk Proc. 3rd IAEA Conf. on Plasma Physics and Controlled Nuclear Fusion, Res. Paper B 10).

and the only non-singular solution $(\xi_r)_{0,0}$ is the trivial solution $(\xi_r)_{0,0}=0$, which implies $\delta W=0$. (ξ_φ and ξ_z are not necessarily zero.) Eqs. (81) and (82) become now:

$$\delta W_{2b}=\lim \delta W_{1a}=\lim \delta W_{2a}=0\,. \qquad (83)$$

Letting $\gamma \rightarrow \infty$ (in accordance with the basic lemma) in Eq. (80) we have

$$\delta W_{2a}=\lim \delta W_{1a}+2\,\pi \int B_\varphi{}^2 \frac{k^2\,r}{k^2\,r^2+m^2}\,\xi_r{}^2\,\mathrm{d}r \geqq \lim \delta W_{1a}\,. \qquad (84)$$

Hence, if we wish to find out whether a pinch (of constant pitch or not) is stable *for incompressible, non-adiabatic perturbations, we need only to examine case 1a* ($k\,r\,B_z+m\,B_\varphi \neq 0$), just as for the adiabatic regime.

Comparison Between the Regimes A and I

As γ does not occur in case 1 a we obtain from Table 2

$$\delta W_{\mathrm{A}}=\delta W_{\mathrm{I}} \qquad (85)$$

where δW is the lowest minimum for both regimes.

As a consequence, we obtain theorem 5 a.

We remind the reader that the reason for the equality (85) is due to the fact that $\operatorname{div}\boldsymbol{\xi}=0$ in both regimes. This is not so in case 2 a: $\operatorname{div}\boldsymbol{\xi}$ is in general non-zero for regime A [see Eq. (79)]. Note however that letting $\gamma \rightarrow \infty$ in Eq. (79) we have $\operatorname{div}\boldsymbol{\xi} \rightarrow 0$ automatically. Putting $\gamma=\infty$ to get the regime I, we obtain from Table 3, for case 2 a:

$$\delta W_{\mathrm{I}}=\delta W_{\mathrm{A}}+2\,\pi \int \frac{B_\varphi{}^2}{B^2+\gamma\,p}\,\frac{\xi_r{}^2}{r}\,\mathrm{d}r \geqq \delta W_{\mathrm{A}}\,. \qquad (86)$$

The inequality (86) holds *in spite of the fact* that $\delta W_{\mathrm{I}}<\delta W_{\mathrm{A}}$ for the unminimized energy integrals with the same perturbation $\boldsymbol{\xi}$ at both sides of the inequality. This gives a good illustration of theorem 3.

We conclude this section 4.3:

Theorem 5.
Comparison of Stability for the Linear Pinch

a. *For a linear pinch, the adiabatic and the incompressible (non-adiabatic) regimes are both stable or both unstable.*

b. *In testing the linear pinch for stability it is sufficient in both the adiabatic and the incompressible (non-adiabatic) regimes to examine perturbations for which* $\operatorname{div}\boldsymbol{\xi}=0$ *and* $k\,r\,B_z+m\,B_\varphi \neq 0$.

Theorem 5 b extends theorem 1 of Newcomb [11] to include the incompressible, non-adiabattic regime and also to include the equilibrium configurations of constant pitch. Theorem 5 results directly from theorem 4 when applied to the linear pinch. Indeed, the linear pinch is one of the examples of an infinitely long tube for which the conditions required for the applicability of the corollary of theorem 4 are fulfilled. The results of this section 4.3 are also summarized in Tables 1 and2.

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